DYNAMICAL APPROXIMATION BY ORBITS OF ERGODIC SYSTEMS

The metric number theory is the field of mathematics which aims at describing finely the size (in terms of Hausdorff dimension or measure) of sets of points approximable at a "certain speed" by a sequence of points of particular interest. The original motivation for such study lies in understanding finely the approximation of real numbers by rationals. The problem formalizes itself in the following fashion: Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a mapping and define

$$E_\psi := \left\{ x \in \mathbb{R} \text{ s.t. } |x - \frac{p}{q}| \leq \psi(q) \text{ i.o. } p, q, p \wedge q = 1 \right\},$$

where *i.o.* means that the inequality holds for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. The definitive answer regarding the dimension of these sets was obtained in 2021 by combining the results obtained in [1, 6]. More precisely, the following result holds true:

$$\dim_H E_\psi \cap [0,1] = \min \{1, s_\psi\},$$

where

$$s_\psi = \sup \left\{ s : \sum_{q \geq 1} \phi(q)\psi(q)^s = +\infty \right\},$$

and $\phi(q) := \# \{0 \leq p \leq q,\ q \wedge p = 1\}$ denotes the Euler mapping.

Although developed in the context of number theory, these approximation results play an important role in various area of mathematics. For instance, given a signal f, the regularity of f around a points t sometimes depends on how well the number t is approximable by certain points naturally connected with the definition of the signal f. As an illustration, if f is a random signal defined using uniform i.i.d sequences, then the mutifractal analysis of f should rely on (almost sure) approximation results of numbers by uniform i.i.d. sequences of random variable. For such reasons (and also because the question is very natural), the approximation theory with respect to random sequences (see [4, 8]) and dynamical sequences (see [3, 7] for instance) have known many developments in the past 20 years. In particular, last year, the following partial results was established in [5].

Theorem 0.1 ([5]). *Let μ be a probability measure on $\mathbb{R}^d$ and $s > \frac{1}{\dim_H \mu}$ then for μ-almost every sequences i.i.d. of law μ $(X_n)_{n \in \mathbb{N}}$, one has*

$$\dim_H \left\{ x : d(x, X_n) \leq \frac{1}{n^s} \ i.o. \ \right\} = \frac{1}{s}$$

In this talk, we will prove that, although the proof of the above result uses the independence of the sequence in an extensive fashion, one can still obtain the same result regarding the approximation of numbers by orbits of dynamical systems. More precisely, we will show the following:

Theorem 0.2. *Let* (T, μ) *be an ergodic dynamical system and* $s > \frac{1}{\dim_H \mu}$ *then for* μ*-almost every* x*, one has*

$$\dim_H \left\{ x : d(x, T^n(x)) \leq \frac{1}{n^s} \ i.o. \ \right\} = \frac{1}{s}.$$

References

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